IJREAT International Journal of Research in Engineering & Advanced Technology, Volume 2, Issue 6, Dec-Jan, 2015
ISSN: 2320 – 8791 (Impact Factor: 1.479)
www.ijreat.org

SEVERAL INTEGRAL CONDITIONS OF OSCILLATION FOR THIRD ORDER LINEAR DIFFERENTIAL EQUATION

JITENDRA KUMAR PATI, DHIRENDRA KUMAR DALAI

ABSTRACT. A generalized oscillation criteria for the equation $y^{'''} + p(t)y^{'} + q(t)y = 0$ with $p(t) \leq 0$, $q(t) \geq 0$ and $p'(t) - q(t) > 0$ where $t \in I$, $I = (a, \infty) \subset (0, \infty)$ is established. At the end, proof of the original Theorem (1.3) by Lazer [10] on oscillation has been revisited taking into account various type of substitutions.

1. INTRODUCTION

Consider the differential equation

$$y^{'''} + p(t)y^{'} + q(t)y = 0 \tag{1.1}$$

where p, q, p' is a mapping from $I \to R$, $I = (a, \infty) \subset (0, \infty)$, $R = (-\infty, \infty)$, are continuous.
Let us take two cases

$$p(t) \leq 0, q(t) > 0, t \in I \tag{1.2}$$

and

$$p(t) \leq 0, p'(t) - q(t) > 0, t \in I \tag{1.3}$$

We consider only nontrivial solution of (1.1). Such a solution is called *oscillatory* on I if it has arbitrarily large zeros, otherwise it is called *non oscillatory* on I.

Equation (1.1)is said to be *oscillatory* on I if it has atleast one *oscillatory* solution.

Consider the differential equation

$$y^{'''} + p(t)y^{'} + q(t)y = 0 \tag{1.4}$$

where p, q, p' is a mapping from $I \to R$, $I = (a, \infty) \subset (0, \infty)$, $R = (-\infty, \infty)$, are continuous.
Let us take two cases

$$p(t) \leq 0, q(t) > 0, t \in I \tag{1.5}$$

and

$$p(t) \leq 0, p'(t) - q(t) > 0, t \in I \tag{1.6}$$

We consider only nontrivial solution of (1.1).

Such a solution is called *oscillatory* on I if it has arbitrarily large zeros, otherwise it is called *non oscillatory* on I. Equation (1.1)is said to be *oscillatory* on I if it has atleast one *oscillatory* solution.

2000 *Mathematics Subject Classification.* 34C10,34C15.
Key words and phrases. Oscillation; non-oscillation; third order differential equations, Integral conditions for oscillation.

Equation(1,1)is said to be of class I on I if and only if every solution y of (1.1) with $y(c) = y'(c) = 0$, $y''(c) > 0$, $c \in [a, \infty]$, has the property that $y(t) > 0$ in (a,c) , in (c, ∞).

Equation(1,1)is said to be of class II on I if and only if every solution y of (1.1) with $y(c) = y'(c) = 0$, $y''(c) > 0$, $c \in [a, \infty]$, has the property that $y(t) > 0$ in (c, ∞).

2. Some useful assertions

The following assertions are helpful in deriving the structure of solutions of equation(1.1). The proofs of these assertions may be omitted since they are similar to proofs in the references.

It may be noted that if y is a solution of equation (1.1) then $-y$ is also the solution of this equation.So concerning *non oscillatory* solutions of (1.1), we restrict our attention only to positive ones.

lem2.1 **Lemma 2.1.** *Let (1.2) hold and y be a nontrivial non oscillatory solution of (1.1). Then there exist $b \geq a$ such that*

$$y(t)y'(t) < 0 \tag{2.1}$$

or

$$y(t)y'(t) \geq 0, y(t) \neq 0 \tag{2.2}$$

for every $t \geq b$. Further more , some positive solution y of type (2.1)satisfies $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$, $y'''(t) < 0$ for all $t \geq a$ and

$$\lim_{t\to\infty} y^{'}(t) = \lim_{t\to\infty} y^{''}(t) = 0, \lim_{t\to\infty} y(t) = k \neq \pm\infty \tag{2.3}$$

Proof. Follow [10; Lemma1.1 ,Lemma 1.3 ,Theorem 1.1],[5; Lemma 2.2]. □

lem2.2 **Lemma 2.2.** *Let (1.2) hold. Then there exist a positive solution y of (1.1) with property (2.1)*

Proof. Follow [10 ; Theorem 1.1]. □

thm2.1 **Theorem 2.3.** *Let (1.2)hold. A necessary and sufficient condition for (1.1) to be oscillatory is that for any non trivial non oscillatory solution y, the condition(2.1)hold.*

Proof. Follow [10; Theorem 1.2]. □

thm2.2 **Theorem 2.4.** *Let (1.2)hold and equation (1.1) be oscillatory. Then any non oscillatory solution y satisfies*

$$\lim_{t\to\infty} y(t) = 0$$

Proof. Follow [8]. □

thm2.3 **Theorem 2.5.** *Let the equation is*

$$y^{'''} + a(t)y^{''} + b(t)y^{'} + c(t)y = 0 \tag{2.4}$$

where

$$a \in C^2([\sigma, \infty), R), b \in C^1([\sigma, \infty), R), c \in C([\sigma, \infty), R), \sigma \in R.$$

The equation is oscillatory if and only if all the non oscillatory solution of the second order differential equation

$$z^{''} + 3zz^{'} + a(t)z^{'} + z^3 + a(t)z^2 + b(t)z + c(t) = 0 \tag{2.5}$$

are eventually negative.

Proof. Suppose that all non oscillatory solution of (2.5) are eventually negative. We have to show that the given equation (2.4) admits an oscillatory solution.

If possible let all solutions of (2.4) be non oscillatory.

So there exist at least one non oscillatory solution $u(t)$ of (2.4) which does not satisfy the condition $u(t)u'(t) < 0$. Without any loss of generality we may take $u(t) > 0$ for $t > t_0 \geq \sigma$. So it follows that $u'(t) \geq 0$ for $t > t_1 > t_0$.

Now taking

$$z(t) = \frac{u'(t)}{u(t)}, t > t_1$$

$u' = zu$ and $u'' = z'u + u'z$, $u''' = z''u + 2z'u' + zu''$, so equation (2.4) becomes

$$z''y + 3zz'y + z^2y' + az'y + az^2y + bzy + cy = 0$$

further it simplifies to

$$z'' + 3zz' + az' = -(z^3 + az^2 + bz + c). \tag{2.6}$$

So $z(t)$ is a nonnegative non oscillatory solution of (2.5), a contradiction.

Hence (2.3) admits an oscillatory solution. Conversely suppose that (2.4)has an oscillatory solution.

If possible let $z(t)$be a positive non oscillatory solution of (2.5). It may be verified that $\vartheta(t) = exp(\int_\sigma^t z(s)ds)$ is a positive increasing solution of (2.4),which is a contradiction.

Hence the proof is completed. □

thm2.4 **Theorem 2.6.** *Suppose that* $a(t){\geq}0$, $b(t){\leq}0$, $c(t){\geq}0$ *and* $a^{'}(t){\leq}0$. *If*

$$\int_\sigma^\infty \left\{\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right\}dt = \infty \tag{2.7}$$

then (2.4) admits oscillatory solutions.

Proof. Let $y(t)$ be a non oscillatory solution of (2.4).

so it follows that there exist a $t_0 \in [\sigma, \infty]$ such that$y'(t) \leq 0$ or ≥ 0 for $t \in [t_0, \infty]$, It is sufficient to prove that $y(t)y'(t) \geq 0$ for $t \geq t_0$ does not hold.

Let$y(t)y'(t) \geq 0$,$t \geq t_0$.

Setting

$$u(t) = \frac{y'(t)}{y(t)}, t \geq t_0$$

we see that $u(t)$ is a solution of the second order Riccati equation

$$z'' + 3zz' + a(t)z' = -F(u(t), t) \tag{2.8}$$

where $F(u(t), t) = u^3(t) + a(t)u^2(t) + b(t)u(t) + c(t)$. It is obvious that $F(u(t), t)$ attains minimum value for $u(t) \geq 0$ at

$$u(t) = \frac{1}{3}\left[-a(t) + \sqrt{(a^2(t) - 3b(t))}\right]$$

The minimum of $F(u(t),t)$ is given by

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}$$

So

$$u''(t)+3u(t)u'(t)+a(t)u'(t) \leq -\left\{\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right\}$$

Integrating the above inequality from t_0 to t we obtain

$$u'(t) \leq u'(t_0) + \frac{3}{2}u^2(t_o) + a(t_0)u(t_0) - \frac{3}{2}u^2(t) - a(t)u(t) + \int_{t_0}^{t} a'(s)u(s)ds$$

$$-\int_{t_0}^{t}\left\{\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s)\right)^{3/2}\right\}ds$$

$$\leq u'(t_0) + \frac{3}{2}u^2(t_o) + a(t_0)u(t_0) - \int_{t_0}^{t}\left\{\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + c(s)\right.$$

$$\left.-\frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s)\right)\right\}ds$$

This in turn implies that

$$\lim_{t\to\infty} u'(t) = -\infty.$$

This completes the Proof of the theorem. □

Definition 2.1. In particular when $p(t) \equiv 0, q(t) > 0, t \in I$, there is the well known oscillation criteria for (1.1) of the form

$$\int^{\infty} t^{2-\epsilon}q(t)dt = \infty$$

for some $\epsilon > 0$.

Definition 2.2. Equation(1.1) is said to have property A if each solution y of this equation is either oscillatory or satisfies condition (2.3).

Remark 2.1. From Theorems 2.1 to 2.4 , it follows that equation(1.1) is oscillatory if and only if it has the property A.

Remark 2.2. We conclude that, in order to prove the equation (1.1) is oscillatory, it is sufficient to prove that (1.1) does not have any non oscillatory solution of type (2.2).

Remark 2.3. By Kneser criterion the basic condition for equation (1.1) to be oscillatory, if $t^2p(t) \leq \frac{1}{4}, t > 0$.

IJREAT International Journal of Research in Engineering & Advanced Technology, Volume 2, Issue 6, Dec-Jan, 2015
ISSN: 2320 – 8791 (Impact Factor: 1.479)
www.ijreat.org

3. Preliminaries

The aim of this paper is to establish several integral criterion for oscillation of equation (1.1) based on the condition (1.2) and (1.3).

Earlier some paper works has been done by [17], [18] with following Remarks and Theorems.

Remark 3.1. If $2q(t) - p'(t) \geq 0$ and not identically zero in any subinterval of I and there exist a number $m < \frac{1}{2}$ such that the second order differential equation $u'' + [p(t) + mtq(t)]u = 0$is oscillatory, then equation (1.1)is oscillatory.

If y is any nonzero solution of (1.1) with $F[y(c)] \leq 0 (c \geq a)$ then y is oscillatory, where $F[y(t)] = 2y(t)y''(t) - (y')^2(t) + p(t)y^2(t)$.

Remark 3.2. Let $0 \leq t^2 p(t) < \frac{1}{4}$ and $q(t) > 0$.

P be the polynomial in the variable z as

$$p(z) = z^3 - 3z^2 + (2 + t^2 p(t))z + t^3 q(t), t > 0$$

Then

$$p(z) \geq t^3 q(t) + t^2 p(t) - \frac{2}{3\sqrt{3}} \left[1 - t^2 p(t)\right]^{\frac{3}{2}}, t > 0 \tag{3.1}$$

for all$z \geq 1 - 2\sqrt{\frac{1-t^2p(t)}{3}}$.

The right hand side of (3.1)is the local minimum of P at the point '

$z_0 = 1 + \sqrt{\frac{1-t^2p(t)}{3}}$.

lem3.1 **Lemma 3.1.** *If $2q(t) - p'(t) \geq 0$ and not identically zero in any subinterval of I and y is a non oscillatory solution of (1.1)which is eventually non negative with $F[y(c)] < 0$, then there exist a number $d \geq c$ such that$y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) \leq 0$ for t≥d.*

Remark 3.3. Any solution y with a zero, that is $y(t^*) = 0$, satisfies $F[y(t^*)] \leq 0$.

thm3.1 **Theorem 3.2.** *Let hypothesis of lemma (3.1) hold and in addition $t^2 p(t) < \frac{1}{4}$ for all $t > 0$. If*

$$\int^{\infty} \left\{ t^2 q(t) + tp(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2 p(t)\right)^{\frac{3}{2}} \right\} dt = \infty \tag{3.2}$$

then equation (1.1)is oscillatory. In fact, any solution y which satisfies $F[y(t^*)] \leq 0$ for some $t^* > a$, is oscillatory.

Proof. Let y be a solution of (1.1)which satisfies $F[y(t_0)] \leq 0$ for some $t_0 > a$. Then by lemma 3.1, y is oscillatory or $y(t)y'(t) > 0$ for all sufficiently large t. suppose without loss of generality that $y(t) > 0$, $y'(t) > 0$ for all $t \geq b \geq t_0$.

Let

$$z(t) = t\frac{y'(t)}{y(t)}, t \geq b$$

so $z(t) > 0$ with $y' = \frac{zy}{t}$, $y'' = \frac{z'y}{t} + \frac{y'z}{t} - \frac{zy}{t^2}$, $y''' = \frac{yz''}{t} + \frac{2y'z'}{t} - \frac{2yz'}{t^2} - \frac{2y'z}{t^2} + \frac{zz'y}{t^2} + \frac{z^2y'}{t^2} - \frac{z^2y}{t^3} + \frac{2yz}{t^3}$.

Hence equation(1.1) becomes

$$\frac{yz''}{t} + \frac{3zz'y}{t^2} - \frac{2z'y}{t^2} - \frac{3z^2y}{t^3} + \frac{z^3y}{t^3} + \frac{2yz}{t^3} + p(t)\frac{zy}{t} + q(t)y = 0.$$

So z satisfies the second order Riccati equation

$$\left\{(tz)' + \frac{3}{2}z^2 - 4z\right\}' + \frac{1}{t}\left\{z^3 - 3z^2 + (2 + t^2p(t))z + t^3q(t)\right\} = 0, \tag{3.3}$$

with the help of Remark 3.2 and Theorem 2.4 we may write

$$\left\{(tz)' + \frac{3}{2}z^2 - 4z\right\}' \leq -\frac{1}{t}\left\{t^3q(t) + t^2p(t) - \frac{2}{3\sqrt{3}}(1 - t^2p(t))^{\frac{3}{2}}\right\} = -Q(t),$$

for all $t \geq b$.

Integrating the above inequality from b to $t \geq b$ we get

$$\left\{tz(t)\right\}' + \frac{3}{2}z^2(t) - 4z(t) \leq k_0 - \int_b^t Q(s)ds$$

where k_o is a constant.

Now $\frac{3}{2}z^2(t) - 4z(t) \geq -\frac{8}{3}$ using $z = 1 - 2\sqrt{\frac{1-t^2p(t)}{3}}$ and $t^2p(t) \leq \frac{1}{4}$.

Integrate the above inequality from b to $t \geq b$ gives

$$tz(t) \leq k_2 + k_1t - \int_b^t\int_b^s Q(u)duds \tag{3.4}$$

where $k_1 = k_0 + \frac{8}{3}$ and $k_2 = b(z(b) - k_1)$.

so it follows from (3.2) and (3.4) that $z(t) < 0$ for sufficiently large t, which contradicts positivity of z.

so equation (1.1) can not have any solution with property $y(t)y'(t) > 0$ for all large t.

By lemma 3.1 equation (1.1) is oscillatory.

This completes the proof of the Theorem. □

thm 3.2

Theorem 3.3. *Let* $0 \leq t^2p(t) < \frac{1}{4}$ *and* $q(t) > 0$, $t \in I$.
If (3.2) is satisfied , then any non oscillatory solution of (1.1) has property

$$\lim_{t\to\infty} y(t) = 0.$$

Proof. Follow [17; Theorem 2].

□

thm3.3

Theorem 3.4. *(Extension of Theorem 3.1) Let (1.2)hold. If*

$$\int^\infty \left\{t^2q(t) + tp(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2p(t)\right)^{\frac{3}{2}}\right\}dt = \infty,$$

then equation(1.1) is oscillatory.

Proof. This is the extension of process as in Theorem 3.1 by taking

$$z = t^2\frac{y'(t}{y(t)}, t \geq b$$

for the completion of the theorem, follow [18,Theorem 3.3]. □

4. Main result

In this section our main aim is to derive several integral conditions regarding oscillation of (1.1) with an extension of conditions and suppositions of [10], [17], [18] by different substitutions.

We have the following Theorems and assumptions which are easily followed.

lem4.1 **Lemma 4.1.** *Let $p(t) \leq 0, q(t) > 0$ hold and Q be the polynomial in the variable z, where*

$$Q(z) = \frac{z^3}{t^{2n}} - \frac{3}{2}n\frac{z^2}{t^{n+1}} + \left(\frac{n(n-1)}{t^2} + p(t)\right)z + q(t)t^n, \quad t > 0.$$

Then

$$Q(z) \geq -\frac{1}{4}n^3t^{3n-3} + \frac{1}{2}n^2(n-1)t^{3n-3} + \frac{1}{2}np(t)t^{3n-1} + q(t)t^{3n}$$

$$-\frac{2}{3\sqrt{3}}t^{3n-3}\left(\frac{3}{4}n^2 - n(n-1) - t^2p(t)\right)^{\frac{3}{2}} = Q(z_0), \tag{4.1}$$

for all $z \geq z_0$ at $z_0 = t^{n-1}\left\{\frac{n}{2} + 3^{-\frac{1}{2}}\left(n - \frac{n^2}{4} - t^2p(t)\right)^{\frac{1}{2}}\right\}$.

Here the right hand side of (4.1) is the local minimum of Q at the point z_0.

Proof. Let

$$Q(z) = \frac{z^3}{t^{2n}} - \frac{3}{2}n\frac{z^2}{t^{n+1}} + \left(\frac{n(n-1)}{t^2} + p(t)\right)z + q(t)t^n, t > 0.$$

Further let

$$F(u(t), t) = u^3(t) + a(t)u^2(t) + b(t)u(t) + c(t) \tag{4.2}$$

with a view to use the concept that the equation $y''' + a(t)y'' + b(t)y' + c(t)y = 0$ is oscillatory if and only if all non oscillatory solutions of the second order differential equation, i.e, Riccati equation

$$z'' + 3zz' + a(t)z' + z^3 + a(t)z^2 + b(t)z + c(t) = 0$$

are eventually negative. Following, $u(t) = \frac{y'(t)}{y(t)}, t \geq t_0$ is a solution of the second order Riccati equation

$$z'' + 3zz' + a(t)z' = -F(u(t), t),$$

where $a(t) = -\frac{3}{2}nt^{n-1}$, $b(t) = \left[\frac{n(n-1)}{t^2} + p(t)\right]t^{2n}$, $c(t) = q(t)t^{3n}$.

$F(u(t), t)$ attains a minimum value at the point

$$u(t) = \frac{1}{3}\left[-a + \sqrt{a^2 - 3b}\right]$$

$$= \frac{1}{3}\left\{\frac{3}{2}nt^{n-1} + \sqrt{\frac{9}{4}n^2t^{2n-2} - 3\left[\frac{n(n-1)}{t^2} + p(t)\right]t^{2n}}\right\}$$

$$= t^{n-1}\left[\frac{n}{2} + 3^{\frac{1}{2}}\left(n - \frac{n^2}{4} - t^2p(t)\right)^{\frac{1}{2}}\right].$$

So the local minimum of Q exists at the point

$$z_0 = t^{n-1}\left[\frac{n}{2} + 3^{\frac{1}{2}}\left(n - \frac{n^2}{4} - t^2 p(t)\right)^{\frac{1}{2}}\right].$$

Now the minimum of $F(u(t), t)$ is given by

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}. \tag{4.3}$$

This becomes

$$-\frac{1}{4}n^3 t^{3n-3} - \frac{1}{3}\left(-\frac{3}{2}n^2(n-1)t^{3n-3} - \frac{3}{2}np(t)t^{3n-1}\right)$$

$$+q(t)t^{3n} - \frac{2}{3\sqrt{3}}\left(\frac{3}{4}n^2 t^{2n-2} - n(n-1)t^{2n-2} - p(t)t^{2n}\right)^{\frac{3}{2}},$$

which simplifies to

$$-\frac{1}{4}n^3 t^{3n-3} + \frac{1}{2}n^2(n-1)t^{3n-3} + \frac{1}{2}np(t)t^{3n-1} + q(t)t^{3n}$$

$$-t^{3n-3}\frac{2}{3\sqrt{3}}\left(\frac{3}{4}n^2 - n(n-1) - t^2 p(t)\right)^{\frac{3}{2}}. \tag{4.4}$$

Next, we show that $Q(z)$ is an increasing function for which $Q(z) \geq Q(z_0)$ at z_0.

$$Q(z) = \frac{z^3}{t^{2n}} - \frac{3}{2}n\frac{z^2}{t^{n+1}} + \left(\frac{n(n-1)}{t^2} + p(t)\right)z + q(t)t^n, t > 0$$

so

$$Q'(z) = \frac{3z^2}{t^{2n}} - \frac{3zn}{t^{n+1}} + \frac{n(n-1)}{t^2} + p(t) = 0. \tag{4.5}$$

Hence

$$t^{2n}Q'(z) = 3z^2 - 3nzt^{n-1} + n(n-1)t^{2n-2} + p(t)t^{2n}.$$

Taking $A = 3, B = -3nt^{n-1}, C = n(n-1)t^{2n-2} + p(t)t^{2n}$,

the discriminant is $D = B^2 - 4AC = t^{2n-2}\left(5n^2 + 4n - 4t^2 p(t)\right)$.

$Q'(z) > 0$ occurs when $t > 0$, $t^{2n-2} > 0$ and $D > 0$, this causes $a > 0$,$b^2 - 4ac < 0$ where $a = 5, b = 4, c = -4t^2 p(t)$.

So $16 + 80t^2 p(t) < 0$ implies $5t^2 p(t) < -1$ which shows that $t^2 p(t) < -\frac{1}{5} < \frac{1}{4}$ satisfies the oscillation criteria.

As $Q'(z) > 0$, so $Q(z)$ is an increasing function. Hence $Q(z) > Q(z_0)$.
So

$$Q(z) \geq -\frac{1}{4}n^3 t^{3n-3} + \frac{1}{2}n^2(n-1)t^{3n-3} + \frac{1}{2}np(t)t^{3n-1} + q(t)t^{3n}$$

$$-\frac{2}{3\sqrt{3}}t^{3n-3}\left(\frac{3}{4}n^2 - n(n-1) - t^2 p(t)\right)^{\frac{3}{2}} = Q(z_0).$$

This completes the proof of the lemma. □

thm4.1 **Theorem 4.2.** *(Extension and generalization of Theorem 3.1 and Theorem 3.3.)* *Let $p(t) \leq 0, q(t) > 0$ hold. If*

$$\int^{\infty} \left\{ \left[\frac{1}{4}n^3 - \frac{1}{2}n^2\right] t^{3n-3} + \frac{1}{2}np(t)t^{3n-1} + q(t)t^{3n} - \frac{2}{3\sqrt{3}}t^{3n-3}\left(n - \frac{n^2}{4} - t^2p(t)\right)^{\frac{3}{2}} \right\} dt = \infty, \tag{4.6}$$

then equation (1.1) is oscillatory for $n \in (2-\sqrt{3}, 2+\sqrt{3})$.

Proof. Let y be non oscillatory solution of (1.1). Suppose without loss of generality that y is positive. We prove that y cannot have the property that $y(t)y'(t) \geq 0$, $y(t) \neq 0$ for every $t \geq b$.

To prove this we assume the contrary, i.e., $y(t) > 0, y'(t) \geq 0, t \geq b \geq a$.

Now we denote

$$z(t) = t^n \frac{y'(t)}{y(t)}, t \geq b.$$

So $z(t) \geq 0$.

Now

$$y' = \frac{yz}{t^n}, y'' = \frac{zy'}{t^n} + \frac{yz'}{t^n} - \frac{nyz}{t^{n+1}}$$

and

$$y''' = \frac{2z'y'}{t^n} + \frac{zy''}{t^n} + \frac{yz''}{t^n} - \frac{2nzy'}{t^{n+1}} - \frac{2nyz'}{t^{n+1}} + \frac{n(n+1)yz}{t^{n+2}}.$$

So equation (1.1) becomes

$$\frac{2z'y'}{t^n} + \frac{z^2}{t^{2n}}y' + \frac{zz'y}{t^{2n}} - \frac{nyz^2}{t^{2n+1}} + \frac{y}{t^n}z'' - \frac{2nzy'}{t^{n+1}} - \frac{2nyz'}{t^{n+1}} + \frac{n(n+1)yz}{t^{n+2}} + p(t)\frac{zy}{t^n} + q(t)y = 0.$$

Thus

$$\frac{3zz'}{t^n} + \frac{z^3}{t^{2n}} - \frac{3nz^2}{t^{n+1}} + z'' - \frac{2nz'}{t} + \frac{n(n+1)z}{t^2} + p(t)z + q(t)t^n = 0.$$

Hence it is easy to verify z satisfies the second order Riccati equation

$$\left(z' + \frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t}\right)' + \frac{z^3}{t^{2n}} - \frac{3}{2}\frac{nz^2}{t^{n+1}} + \left(\frac{n(n-1)}{t^2} + p(t)\right)z + q(t)t^n = 0. \tag{4.7}$$

By Lemma 4.1 we have

$$\left(z' + \frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t}\right)' \leq -\left[-\frac{1}{4}n^3t^{3n-3} + \frac{1}{2}n^2(n-1)t^{3n-3} + \frac{1}{2}np(t)t^{3n-1} + q(t)t^{3n} - t^{3n-3}\frac{2}{3\sqrt{3}}\left(\frac{3}{4}n^2 - n(n-1) - t^2p(t)\right)^{\frac{3}{2}}\right] = -Q(z_0)$$

for all $t \geq b$.

Integrating the above inequality from b to $t \geq b$, we get

$$\left(z' + \frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t}\right) \leq k_0 - \int_b^t Q[z_0(s)]ds, \tag{4.8}$$

where k_0 is a constant.

Now

$$\frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t} = \left(t^{-n}z(t)\right) \times \left(\frac{3}{2}z(t) - 2nt^{n-1}\right)$$

$$\frac{3}{2}z(t) - 2nt^{n-1} = \frac{3}{2}t^{n-1}\left[\frac{n}{2} - \frac{4n}{3} + \sqrt{3}\sqrt{n - \frac{n^2}{4} - t^2p(t)}\right].$$

And

$$\sqrt{n - \frac{n^2}{4} - t^2p(t)} > \sqrt{-\frac{1}{4} + n - \frac{n^2}{4}} = \sqrt{n - \frac{n^2+1}{4}},$$

as $t^2p(t) \leq \frac{1}{4}$.

So

$$\frac{3}{2}z(t) - 2nt^{n-1} \geq \frac{3}{2}t^{n-1}\left\{\sqrt{3}\left(\sqrt{n - \frac{n^2+1}{4}}\right) - \frac{5n}{6}\right\}.$$

Hence

$$\left(t^{-n}z(t)\right) \times \left(\frac{3}{2}z(t) - 2nt^{n-1}\right) \geq t^{-1}\left[\frac{n}{2}\right.$$

$$\left.+\sqrt{3}\sqrt{n - \frac{1}{4}(n^2+1)}\right] \times \frac{3}{2}t^{n-1}\left[-\frac{5n}{6} + \sqrt{3}\sqrt{n - \frac{n^2+1}{4}}\right]$$

$$= \frac{3}{2}t^{n-2}\left\{-\frac{7}{6}n^2 + 3n - \frac{3}{4} - \frac{n}{\sqrt{3}}\sqrt{n - \frac{n^2+1}{4}}\right\}.$$

For the sake of our convenience it may be supposed that

$$\frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t} \geq \frac{3}{2}t^{n-2}\left\{-\frac{7}{6}n^2 + 3n - \frac{3}{4} - \frac{n}{\sqrt{3}}\sqrt{n - \frac{n^2+1}{4}}\right\}$$

In particular for $n = 2$, the value of the right hand side is $-\frac{5}{12} \geq -\frac{8}{3}$.
So

$$\frac{3}{2}\frac{z^2}{t^n} - \frac{2nz}{t} \geq -\frac{8}{3}$$

with the condition $n - \frac{1}{4}(n^2+1) > 0$ i.e., $|n-2| < \sqrt{3}$ or $2 - \sqrt{3} < n < 2 + \sqrt{3}$.
Integrating both sides of the inequality (4.8) again from b to $t \geq b$, we get

$$z(t) \leq k_1 + k_2t - \int_b^t \int_b^s Q\Big(z_0(u)\Big)duds \tag{4.9}$$

where

$$k_1 = z(b) + \frac{8}{3}b - k_0b, k_2 = k_0 + \frac{8}{3}.$$

So following (4.6) and (4.9), we obtain that $z < 0$ for sufficiently large t.

This contradicts the fact that z is non negative. Therefore equation (1.1) can not have any solution with the assumed property.

We get a proof of the theorem.

□

thm4.2 **Theorem 4.3.** *Let (1.3)hold. If*

$$\int^{\infty}\left\{\left[\frac{1}{4}n^3 - \frac{1}{2}n^2\right]t^{3n-3} + \frac{1}{2}np(t)t^{3n-1}\right.$$

$$\left.+\Big[p'(t) - q(t)\Big]t^{3n} - \frac{2}{3\sqrt{3}}t^{3n-3}\left[n - \frac{n^2}{4} - t^2p(t)\right]^{\frac{3}{2}}\right\}dt = \infty$$

then equation (1.1)is oscillatory.

Proof. By condition(1.3) we have $p'(t) - q(t) > 0$,
so the equivalent equation of (1.1) under this condition is

$$y''' + p(t)y' + \left[p'(t) - q(t)\right]y = 0. \tag{4.10}$$

Now by results of Hanan [7, theorem 3.3, lemma 2.9], equation(4.10) is of class$I$, so equation (1.1)is of class II.

hence, by [7;Theorem 4.7], equation(1.1)is oscillatory if and only if equation (4.10) is oscillatory. So, applying Theorem 4.1 to equation (4.10), we obtain proof of the theorem. □

thm4.3 **Theorem 4.4.** *Let (1.2) hold. If*

$$\int^{\infty}\left[q(t) - \frac{2}{3\sqrt{3}}\Big(-p(t)\Big)^{\frac{3}{2}}\right]dt = \infty$$

Then equation (1.1) is oscillatory

Proof. The proof of the Theorem may be followed with an important observation. □

Important Observation. Earlier by Lazer [10;Theorem1.3] the theorem has been proved by using certain substitution for z.

Here we are focusing on the extension of the theorem with different substitutions for $z(t)$ to get the second order Riccati equation, which yields the integral condition for oscillation of (1.1)

Proof. Let us consider the equation

$$y''' + ay'' + by' + cy = 0 \tag{4.11}$$

compare with original equation(1.1) we have $a = 0$, $b = p(t)$, $c = q(t)$. Let y be the solution of(4.11) and equivalently solution of (1.1) with an assumption that y is any non oscillatory solution.

Suppose without loss of generality y is positive. We prove that y cannot have the property (2.2). To prove this we assume the contrary i.e $y(t) > 0$, $y'(t) \geq 0$ for $t \geq b \geq a$.

Let we denote

$$z(t) = e^t \frac{y'}{y}$$

So $z > 0$ as $t > 0$

By this

$$y' = \frac{yz}{e^t}$$

,

$$y'' = \frac{y'z}{e^t} + \frac{yz'}{e^t} - \frac{yz}{e^t}$$

$$y''' = \frac{y''z}{e^t} + \frac{2y'z'}{e^t} - \frac{2yz'}{e^t} + \frac{yz''}{e^t} + \frac{yz}{e^t} - \frac{2y'z}{e^t}$$

Hence (4.11) yields

$$\frac{z''}{e^t} + \frac{3zz'}{e^{2t}} + \frac{(a-2)z'}{e^t} =$$

$$-\left\{\frac{z^3}{e^{3t}} + \frac{(a-3)z^2}{e^{2t}} + \frac{(1-a+b)z}{e^t} + c\right\}$$

Letting $\frac{z}{e^t} = \alpha$, we have the above equation becomes

$$\alpha'' + 3\alpha\alpha' + a\alpha' = -(\alpha^3 + a\alpha^2 + b\alpha + c)$$

So $\alpha(t)$ satisfies the second order nonlinear Riccati equation. Here $a = 0$,$b = p(t)$, $c = q(t)$ and $\alpha(t) > 0$ Hence

$$\alpha'' + 3\alpha\alpha' = -\Big(\alpha^3 + p(t)\alpha + q(t)\Big) \tag{4.12}$$

Letting

$$F\Big(\alpha(t), t\Big) = \alpha^3 + p(t)\alpha + q(t)$$

we have by (2.8), the minimum of the function $F\Big(\alpha(t), t\Big)$ is given by

$$\frac{2a^3}{27} - \frac{ab}{3} + c - \frac{2}{3\sqrt{3}}(\frac{a^2}{3} - b)^{\frac{3}{2}}$$

which yields

$$q(t) - \frac{2}{3\sqrt{3}}\Big[-p(t)\Big]^{\frac{3}{2}}$$

So from (4.12) we obtain

$$\frac{d}{dt}\Big(\alpha' + \frac{3}{2}\alpha^2(t)\Big) \leq -q(t) + \frac{2}{3\sqrt{3}}\Big(-p(t)\Big)\frac{3}{2} \tag{4.13}$$

Integrate the inequality (4.13)both sides from b to $t \geq b$ we have

$$\alpha'(t) \leq \alpha'(b) + \frac{3}{2}\alpha^2(b) - \frac{3}{2}\alpha^2(t)$$
$$- \int_b^t \Big[q(t) - \frac{2}{3\sqrt{3}}\Big(-p(t)\Big)^{\frac{3}{2}}\Big]dt$$

So

$$\alpha'(t) \to -\infty$$

as $t \to +\infty$ and by given condition.

Consequently $\alpha(t)$ would eventually become negative, which contradicts the assumption that $\alpha(t)$ is positive and $y(t)y'(t) \geq 0$ for $t \geq b$.

So (1.1) is oscillatory, since the second order Riccati equation does not admit a non oscillatory solution that is eventually positive by referring Theorem (2.3). □

Example 4.1. Consider the differential equation

$$y''' - (1 - e^{-t})y' + (\frac{2}{3\sqrt{3}} + b)y = 0, b = 0 \tag{4.14}$$

Here $p(t) = -(1 - e^{-t}) < 0$, $q(t) = \frac{2}{3\sqrt{3}} + b > 0$.

So

$$\int_0^\infty \Big[q(t) - \frac{2}{3\sqrt{3}}\Big(-p(t)\Big)^{\frac{3}{2}}\Big]dt$$
$$= \int_0^\infty \Big[\frac{2}{3\sqrt{3}} + b - \frac{2}{3\sqrt{3}}(1 - e^{-t})^{\frac{3}{2}}\Big]dt$$
$$= \int_0^\infty \Big[\frac{2}{3\sqrt{3}} + b - \frac{2}{3\sqrt{3}}\Big(1 + \frac{3}{2}e^{-t} - \frac{3}{8}e^{-2t} + - - -\Big)\Big]dt = +\infty$$

So by the Theorem (4.3),
the equation has oscillatory solution.

Remark 4.1. Following the proof of the theorem 4.3,
we may observe the extension of the theorem 4.3 by more different substitutions as follows.

thm4.4 **Theorem 4.5.** *Let (1.1) hold.If*

$$\int^{\infty}\left[q(t)-\frac{2}{3\sqrt{3}}\Big(-p(t)\Big)^{\frac{3}{2}}\right]dt=\infty$$

Then equation (1.1) is oscillatory.

Proof. In this case only we show how the equation (1.1) will be modified to a second order non linear Riccati equation by taking different substitutions. Remaining steps for the proof are easily followed from Theorem (4.3).

Substitution 1. Let $z(t)=e^{-t}\frac{y'}{y}$, obviously $z>0$
With this $y'=yze^{t}$, $y''=y'ze^{t}+yz'e^{t}+yze^{t}$, $y'''=y'ze^{2t}+yzz'e^{2t}+yz^{2}e^{2t}+2y'z'e^{t}+2y'ze^{t}+yz''e^{t}+2yz'e^{t}+yze^{t}$
So equation(1.1)becomes

$$z^{3}e^{3t}+3zz'e^{2t}+3z^{2}e^{2t}+z''e^{t}+2z'e^{t}+(1+p)ze^{t}+q=0$$

By letting $ze^{t}=\alpha$ we find the relation

$$\alpha^{3}+3\alpha\alpha'+\alpha''+p\alpha+q=0. \tag{4.15}$$

Substitution 2. Let $z(t)=e^{it}\frac{y'}{y}$ obviously $z>0$
By this $y'=\frac{yz}{e^{it}}$, $y''=\frac{y'z}{e^{it}}+\frac{yz'}{e^{it}}-\frac{iyz}{e^{it}}$ and $y'''=\frac{z^{3}y}{e^{3it}}+\frac{3yzz'}{e^{2it}}-\frac{3iyz^{2}}{e^{2it}}-\frac{2iyz'}{e^{it}}+\frac{yz''}{e^{it}}-\frac{yz}{e^{it}}$
So equation (1.1) becomes

$$\frac{z^{3}}{e^{3it}}+\frac{3zz'}{e^{2it}}-\frac{3iz^{2}}{e^{2it}}-\frac{2iz'}{e^{it}}+\frac{z''}{e^{it}}-\frac{z}{e^{it}}+p(t)\frac{z}{e^{it}}+q(t)=0$$

By letting $\frac{z}{e^{it}}=\alpha$
we find the relation

$$\alpha^{3}+3\alpha\alpha'+\alpha''+p\alpha+q=0. \tag{4.16}$$

Substitution 3. Let $z(t)=cost\frac{y'}{y}=\left(\frac{e^{it}+e^{-it}}{2}\right)\frac{y'}{y}$
It is obvious $z>0$
So $y'=yzsect$, $y''=y'zsect+yz'sect+yzsecttant$ and $y'''=y'z^{2}sec^{2}t+3yzz'sec^{2}t+3yz^{2}sec^{2}ttant+yz''sect+2yz'secttant+yzsec^{2}ttant+yzsec^{3}t$
So equation(1.1) becomes

$$z^{3}sec^{3}t+3zz'sec^{2}t+3z^{2}sec^{2}ttant+z''sect+2z'secttant+zsec^{2}ttant+zsec^{3}t+pzsect+q=0$$

By letting $zsect=\alpha$ we also get the relation

$$\alpha^{3}+3\alpha\alpha'+\alpha''+p\alpha+q=0 \tag{4.17}$$

Similarly we may try for different substitutions as $z=sint\frac{y'}{y}$,$sinht\frac{y'}{y}$,$cosht\frac{y'}{y}$etc.
After computing the relation (4.15), (4.16), (4.17) we may proceed for the proof of the theorem 4.4 by referring the procedure of theorem 4.3. □

thm4.5 **Theorem 4.6.** *Let (1.3)hold.If*

$$\int^{\infty}\left[\left(p'(t)-q(t)\right)-\frac{2}{3\sqrt{3}}\left(-p(t)\right)\frac{3}{2}\right]dt=\infty \tag{4.18}$$

Then equation(1.1)is oscillatory.

Proof. For proof follow the theorem 4.2 under the condition (1.3). □

Remark 4.2. Let $p(t)=0$ and $q(t)>0$ for $t\in I$.
So

$$\frac{1}{4}n^3t^{3n-3}-\frac{1}{2}n^2t^{3n-3}\geq\frac{2}{3\sqrt{3}}t^{3n-3}.$$

Then

$$\frac{n^3-2n^2}{4}\geq\frac{2}{3\sqrt{3}}.$$

This yields

$$q(t)t^{3n}\geq\frac{2}{3\sqrt{3}}t^{3n-3}$$

or

$$t^3q(t)\geq\frac{2}{3\sqrt{3}}>\frac{1}{3\sqrt{3}}.$$

So $y'''+q(t)y=0, q(t)>0$, is oscillatory if

$$liminf\, t^3q(t)>\frac{2}{3\sqrt{3}} \tag{4.19}$$

References

c1 [1] L.Cecchi and M. Marini ; On the oscillatory behavior of a third order nonlinear differential equation,
Nonlinear Anal., **15** (1990),141 - 153.

c2 [2] T.A.Chanturiya; Integral conditions of oscillation of solutions of higher order linear differential equations,
Differentsial'nye Uravneniya., **16** (1980),470 - 482.

d1 [3] J. Dzurina ; Comparison theorems for nonlinear ODEs, *Math. Slovaca.*, **42** (1992),299 - 315.

e1 [4] J. Elias ; Properties of the nonoscillatory solutions for a third order nonlinear diferential equation, *Mat.Casopis.*, **20** (1970),249 - 253.

e2 [5] L.Erbe ; Exixtence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations, *Pacific J. Math.*, **64** (1976),369 - 385.

e3 [6] L.Erbe ; Oscillation, nonoscillation and asymptotic behavior for third order nonlinear differential equations, *Ann.Mat. Pura Appl.(4).*, **110** (1976),373 - 391.

h1 [7] M.Hanan ; Oscillation criteria for a third order linear differential equations, *Pacific J. Math.*, **11** (1961), 919 - 944.

j1 [8] G.D.Jones ; An asymptotic property of solutions $y'''+p(x)y'+q(x)y=0$, *Pacific J. Math.*, **17** (1973), 135 - 138.

k1 [9] N.N.Khvedelidze and T.A. Chanturiya ; Oscillation of solutions of third order linear ODEs(Russian), *Differentsial"nye Uravneniya.*, **27** (1991), 452 - 460, 611-618.

l1 [10] A.C.Lazer ; The behavior of solutions of the differential equation $y'''+p(x)y'+q(x)y=0$, *Pacific J. Math.*, **17** (1966), 435 - 466.

l2 [11] L.Licko and M.Svec ; La caractere oscillatoire des solutions de L'equation $y^n+f(t)y^{\alpha}=0, n>1$, *Czechoslovak Math.J.*, **13** (1963), 481 - 491.

n1 [12] J.L. Nelson ; A stability theorem for a third order nonlinear differential equation, *Pacific J. Math.*, **24** (1968), 341 - 344.

o1 [13] R. Olah ; Integral conditions of oscillation of a linear differential equation, *Math.Slovaca.*, **39** (1989), 323 - 329.

r1 [14] J.Rovder ; Oscillation criteria for third order linear differential equation, *Mat.Casopis.*, **25** (1975), 231 - 244.

s1 [15] A.Skerlik ; Criteria of property A for third order superlinear differential equations, *Math.Slovaca.*, **45** (1993), 171 - 183.

s2 [16] P.Soltes ; A remark on the oscillatoriness of solutions of a nonlinear third order equation, *Mat.Casopis.*, **23** (1973), 326 - 332.

s3 [17] A.Skerlik ; An Integral condition of oscillation for equation $y''' + p(t)y' + q(t)y = 0$ with nonnegative coefficients, *Archivum mathematicum(brno).*, **31** (1995), 155 - 161.

s3 [18] A.Skerlik ; Integral criteria of oscillation for a third order linear differential equation, *Mathematica Slovaca.*, **45** (1995), 403 - 412.

p1 [19] P.Das and J.K.Pati ; Necessary and Sufficient condition for oscillation of a third order differential equation, *Electronic Journal of Differential Equations.*, **174** (2010), 1 - 10.

p1 [20] J.K.Pati and D.K.Dalai ; Conditions for Oscillatory solutions of a third order linear differential equations, *American Journal of Applied Mathematics and Mathematical Sciences.*, **2** (2013), 91 - 96.

p1 [21] J.K.Pati and D.K.Dalai ; Oscillation Criteria for third order Nonlinear and Non Homogenous differential equation, *International j.of Math.Sci. and Engg. Appls.*, **7** (2013), 289 - 296.

p1 [22] J.K.Pati and D.K.Dalai and R.B.Dash ; Oscillatory and Non oscillatory behavior of the equation $y''' + p_0t^{\beta}y' + q_0t^{\delta}y = 0$, *Journal of the Indian math. soc.*, **81** (2014), 295 - 308.

Jitendra Kumar Pati
Department of Mathematics, Synergy Institute of Engg and Technology,, Dhenkanal, Odisha, India
E-mail address: `jkpati2007@yahoo.com`

Dhirendra Kumar Dalai
Department of Mathematics, S.B.Women's College,, Cuttack, Odisha, India
E-mail address: `drdalai@yahoo.com`